Extra MR Problems

DTC Medical Imaging course
April, 2014

The problems below are harder, more time-consuming, and intended for those with a more mathematical background. They are **entirely optional**, but hopefully will make the mathematicians happy.

1 Spin physics

d) * By considering two spin states of energy E_1 and E_2, derive the Boltzmann equation for the ratio of spins in each state. *Answer: There are many ways of doing this, and this answer is by no means the best. The Maxwell-Boltzmann distribution $f(E)$ gives you the probability distribution of energy in the system: $f(E) \propto \Omega(E)e^{-E/k_BT}$ where $\Omega(E)$ is the degeneracy of state E. We expect the degeneracy to be the same for each of the two states. Therefore, the ratio of populations is going to be proportional to the ratio of these two distributions, i.e.*

$$\frac{f(E_1)}{f(E_2)} = \frac{n_\uparrow}{n_\downarrow} = \frac{e^{-E_1/k_BT}}{e^{-E_2/k_BT}} = e^{-\frac{\Delta E}{k_BT}}$$

e) * Deduce how the spin state population ratios change with B in the regime where $\Delta E/k_BT$ is small, i.e. find $\frac{\partial}{\partial B}$. How do you think this compares to the change in cost of an MR system with magnetic field? *Answer: In the given regime, we are able to Taylor expand the exponential function, so that the ratio is approximately $1 - \frac{\hbar\gamma B_0}{k_BT}$. We can trivially differentiate with respect to B, and come to the conclusion that signal is improved – i.e. the ratio decreased from unity – approximately linearly with B. (In fact, it's slightly worse than this – the sign of the next largest B^2 term is going to be positive, which reduces the signal we see). It is highly likely, however, that the cost of an MR system does not scale linearly with B.*

2 Radiofrequency pulses and hardware

e) ** *[Nutters only]* The way to understand, in general, the effect of an RF pulse is to solve the Bloch equations. These describe the evolution of magnetisation in matter, and are a phenomenological extension to a result that arises out of Schrödinger equations. Neglecting T_1 and T_2, the Bloch equations can be written in the rotating reference frame as

$$\begin{pmatrix} \frac{\mathrm{d}M_x(t)}{\mathrm{d}t} \\ \frac{\mathrm{d}M_y(t)}{\mathrm{d}t} \\ \frac{\mathrm{d}M_z(t)}{\mathrm{d}t} \end{pmatrix} = \gamma \begin{pmatrix} 0 & \mathbf{G}\cdot\mathbf{x} & -B_{1,y} \\ -\mathbf{G}\cdot\mathbf{x} & 0 & B_{1,x} \\ B_{1,y} & -B_{1,x} & 0 \end{pmatrix} \begin{pmatrix} M_x \\ M_y \\ M_z \end{pmatrix} \tag{1}$$

where $\mathbf{G} = (G_x, G_y, G_z)$ is an applied linear magnetic field gradient, $\mathbf{x} = (x(t), y(t), z(t))$, and B_1 the applied RF field. Both $\mathbf{G}$ and B_1 are functions of time. We now consider a small flip angle pulse, such that $M_z \approx M_0 \approx$ constant.

By defining the transverse magnetisation $M_{xy} = M_x + iM_y$, and the applied field $B_1 = B_{1,x} + iB_{1,y}$, show that the first two components of (1) can be written as

$$\frac{\mathrm{d}M_{xy}}{\mathrm{d}t} = -i\gamma\mathbf{G}\cdot\mathbf{x}M_{xy} + i\gamma B_1 M_0.$$

Answer: Multiply the second equation by i and add the two:

$$\begin{aligned} \frac{\partial}{\partial t}(M_x + iM_y) &= \gamma\mathbf{G}\cdot\mathbf{x}M_y - \gamma B_{1,y}M_0 - i\gamma\mathbf{G}\cdot\mathbf{x}M_x + i\gamma B_{1,x}M_0 \\ &= \gamma\mathbf{G}\cdot\mathbf{x}(M_y - iM_x) + \gamma M_0(-B_{1,y} + iB_{1,x}) \\ &= \gamma\mathbf{G}\cdot\mathbf{x}(-i^2M_y - iM_x) + \gamma M_0(i^2B_{1,y} + iB_{1,x}) \\ &= \gamma\mathbf{G}\cdot\mathbf{x}(-i)(M_x + iM_y) + \gamma M_0 i(B_{1,x} + iB_{1,y}) \\ \Rightarrow \frac{\partial}{\partial t}M_{xy} &= -i\gamma\mathbf{G}\cdot\mathbf{x}M_{xy} + iM_0\gamma B_1 \end{aligned}$$

Now, if the system is initially in the state $\mathbf{M} = (0, 0, M_0)$, show that this differential equation can be solved for the final magnetisation at a time T to yield

$$M_{xy}(\mathbf{x}) = i\gamma M_0 \int_0^T B_1(t)e^{-i\gamma\cdot\int_t^T \mathbf{G}(s)\cdot x(s)\,\mathrm{d}s}\,\mathrm{d}t. \tag{2}$$

Answer: This is a simple first-order ODE, and just requires an integrating factor. The appropriate integrating factor is $e^{i\int_0^T \gamma G(s)\cdot x\,\mathrm{d}s}$ – see, e.g, physics or engineering first year textbooks (Riley, Hobson & Bence or Kryzig) for a derivation. So, multiply both sides of the DE by this integrating factor, and the left side is an exact derivative. We therefore have:

$$\frac{\partial}{\partial t}\left[M_{xy}e^{i\int_0^T \gamma(s)\,\mathrm{d}s}\right] = i\gamma M_0 B_1(t)e^{i\int_0^T \gamma G(s)\cdot x\,\mathrm{d}s}.$$ *Integrating:*

$$\begin{aligned} M_{xy}e^{i\int_0^T \gamma(s)\,\mathrm{d}s} &= iM_0\int_0^T \gamma B_1(t)e^{i\int_0^T \gamma G(s)\cdot x\,\mathrm{d}s}\,\mathrm{d}t \\ M_{xy} &= iM_0 e^{-i\int_0^T \gamma(s)\,\mathrm{d}s}\int_0^T \gamma B_1(t)e^{i\int_0^T \gamma G(s)\cdot x\,\mathrm{d}s}\,\mathrm{d}t \\ &= iM_0\int_0^T \gamma B_1(t)e^{-i\int_0^t \gamma G(s)\cdot x\,\mathrm{d}s}e^{i\int_0^T \gamma G(s)\cdot x\,\mathrm{d}s}\,\mathrm{d}t. \text{ Combining limits:} \\ &= iM_0\int_0^T \gamma B_1(t)e^{-i\int_t^T \gamma G(s)\cdot x\,\mathrm{d}s}\,\mathrm{d}t \end{aligned}$$

If we now define a spatial frequency variable $\mathbf{k}(t) = -\gamma\int_t^T \mathbf{G}(s)\,\mathrm{d}s$, we can re-write (2) as something more pleasant. By writing the exponential factor as an integral over a three-dimensional delta function, interchanging the order of integration, and defining a new function

$$p(\mathbf{k}) = \int_0^T B_1(t)\delta^3(\mathbf{k}(t) - \mathbf{k})\,\mathrm{d}t$$

show that the magnetisation response to an RF pulse is therefore

$$M_{xy}(\mathbf{x}) = i\gamma M_0 \int_{\mathbf{K}} p(\mathbf{k})e^{i\mathbf{x}\cdot\mathbf{k}}\,\mathrm{d}\mathbf{k} \tag{3}$$

and state what this tells you about the relationship between an RF pulse and its effect on matter. *Answer: Let us define the spatial frequency variable* $\mathbf{k}(t)$ *by*

$$\mathbf{k}(t) = -\gamma \int_t^T \mathbf{G}(s)\,\mathrm{d}s$$

then we can re-write things in terms of k*:*

$$M_{xy} = i\gamma M_0 \int_0^T B_1(t)e^{i\mathbf{k}(\mathbf{t})\cdot\mathbf{x}}\,\mathrm{d}t$$

Now the function k *parametrically describes a path through spatial frequency space. We can express the exponential as an integral over* k *space by using the Dirac delta function, which, as you will recall, lets us make substitutions like* $f(a) = \int f(x)\delta(x-a)\,\mathrm{d}x$*:*

$$M_{xy} = i\gamma M_0 \int_0^T B_1(t) \int_{\mathbf{K}} \delta^3(\mathbf{k}(t) - \mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}\,\mathrm{d}\mathbf{k}\,\mathrm{d}t$$

We can now interchange the order of integration freely (believe it or not, the delta function is differentiable and continuous):

$$M_{xy} = i\gamma M_0 \int_{\mathbf{K}} \left\{ \int_0^T B_1(t)\delta^3(\mathbf{k}(t) - \mathbf{k})\,\mathrm{d}t \right\} e^{i\mathbf{k}\cdot\mathbf{x}}\,\mathrm{d}\mathbf{k}$$

and define the function

$$p(\mathbf{k}) = \int_0^T B_1(t)\delta^3(\mathbf{k}(t) - \mathbf{k})\,\mathrm{d}t$$

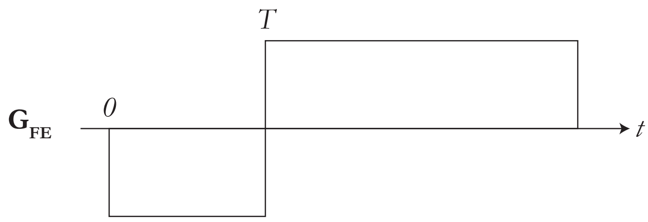
such that the overall response is

$$M_{xy}(\mathbf{x}) = i\gamma M_0 \int_{\mathbf{K}} p(\mathbf{k})e^{i\mathbf{x}\cdot\mathbf{k}}\,\mathrm{d}\mathbf{k}.$$

This makes the Fourier relationship really bloody obvious, and we have derived the excitation k*-space formalism. Woo. The function* p *shows the explicit weighting of excitation* k*-space by the RF excitation* $B_1(t)$ *and the trajectory taken.*

5 Frequency Encoding Gradients

e) ** Consider a series of spins arranged on a one-dimensional lattice at locations x_i. Let the density of spins at each site be ρ_i. We are now going to explore what happens during a prephasing gradient and under a frequency encoding gradient, shown schematically below.



i) When a prephasing gradient $G_{FE,p}(t)$ is played, what will the precession frequency at each site be? *Answer:* $\gamma(B_0 + G_{FE,P}(t)x_i)$

ii) In the rotating reference frame, show that the phase accrued by spin i due to this gradient is given by

$$\phi_p(x_i, T) = \gamma x_i \int_0^T G_{FE,p}(t)\,\mathrm{d}t \equiv 2\pi x_i k_{x,p}$$

where T is the duration of the prephasing gradient. Define $k_{x,p}$. *Answer: In the rotating reference frame B_0 vanishes from the above equation. We know that $\omega = \frac{\mathrm{d}\phi}{\mathrm{d}t}$, so it then naturally follows that*

$$\phi_p(x_i, T) = \gamma x_i \int_0^T G_{FE,p}(t)\,\mathrm{d}t \equiv 2\pi x_i k_{x,p}$$

with $k_{x,p} = (\gamma/2\pi)\int_0^T G_{FE,p}(t)\,\mathrm{d}t$ being the k-space offset caused by the prephasing gradient.

iii) The NMR signal $S_{x,p}$ observed is the sum of spin densities, with weights given by the phase. Write down this sum. *Answer:*

$$S_{x,p} = \sum_{j=1}^{n} \rho_{x_j} e^{-i\phi_p(x_i,T)}$$

iv) If we let the number of spins become continuous, this sum generalises to

$$S_{x,p} = \int_{-\infty}^{\infty} \rho(x) e^{i\phi_p(x,T)}\,\mathrm{d}x$$

where $\rho(x)$ and $\phi_p(x, T)$ are the continuous analogies of those functions defined as above. Let us now see what happens when we play a frequency encoding gradient lobe $G_{FE}(t)$ at this point in time. Show that the NMR signal as a function of time becomes

$$S(t) = \int_{-\infty}^{\infty} \rho(x) e^{i(\phi(x,t) - \phi_p(x,T))} \, \mathrm{d}x \tag{4}$$

with $\phi(x, t)$ defined analogously for the frequency encoding gradient as to the prephasing gradient. *Answer: The only difference is that now we start acquiring phase (at an increasing rate) in the other direction, caused by the readout gradient. Everything is exactly the same, except that how much phase we've got is a function of time. This leads instantly to the equation given.*

v) What happens when $(\phi(x, t) - \phi_p(x, T)) = 0$? *Answer: Everything is in phase, and the signal is just the integral of spin density – in other words, we have an echo.*

vi) As the prephasing and frequency encoding gradients have opposite signs, show that this happens at a time t_{echo} defined by

$$-\int_0^T G_{FE,p}(t) \, \mathrm{d}t = \int_0^{t_{\text{echo}}} G_{FE}(t) \, \mathrm{d}t$$

Answer: At t_{echo}, the requirement is for $(\phi(x, t) - \phi_p(x)) = 0$. Going back to what actually generates the phases, we see that this is just the same thing as

$$-\int_0^T G_{FE,p}(t) \, \mathrm{d}t = \int_0^{t_{echo}} G_{FE}(t) \, \mathrm{d}t$$

This defines the relationship between echo time and the area under gradient lobes. Additionally, (4) illustrates that the observed signal is related to the Fourier transform of spin density.