Extra MR Problems

DTC Medical Imaging course
April, 2014

The problems below are harder, more time-consuming, and intended for those with a more mathematical background. They are **entirely optional**, but hopefully will make the mathematicians happy.

1 Spin physics

d) * By considering two spin states of energy E_1 and E_2, derive the Boltzmann equation for the ratio of spins in each state.

e) * Deduce how the spin state population ratios change with B in the regime where $\Delta E/k_B T$ is small, i.e. find $\frac{\partial}{\partial B}$. How do you think this compares to the change in cost of an MR system with magnetic field?

2 Radiofrequency pulses and hardware

e) ** *[Nutters only]* The way to understand, in general, the effect of an RF pulse is to solve the Bloch equations. These describe the evolution of magnetisation in matter, and are a phenomenological extension to a result that arises out of Schrödinger equations. Neglecting T_1 and T_2, the Bloch equations can be written in the rotating reference frame as

$$\begin{pmatrix} \frac{\mathrm{d}M_x(t)}{\mathrm{d}t} \\ \frac{\mathrm{d}M_y(t)}{\mathrm{d}t} \\ \frac{\mathrm{d}M_z(t)}{\mathrm{d}t} \end{pmatrix} = \gamma \begin{pmatrix} 0 & \mathbf{G}\cdot\mathbf{x} & -B_{1,y} \\ -\mathbf{G}\cdot\mathbf{x} & 0 & B_{1,x} \\ B_{1,y} & -B_{1,x} & 0 \end{pmatrix} \begin{pmatrix} M_x \\ M_y \\ M_z \end{pmatrix} \tag{1}$$

where $\mathbf{G} = (G_x, G_y, G_z)$ is an applied linear magnetic field gradient, $\mathbf{x} = (x(t), y(t), z(t))$, and B_1 the applied RF field. Both $\mathbf{G}$ and B_1 are functions of time. We now consider a small flip angle pulse, such that $M_z \approx M_0 \approx$ constant.

By defining the transverse magnetisation $M_{xy} = M_x + iM_y$, and the applied field $B_1 = B_{1,x} + iB_{1,y}$, show that the first two components of (1) can be written as

$$\frac{\mathrm{d}M_{xy}}{\mathrm{d}t} = -i\gamma\mathbf{G}\cdot\mathbf{x}M_{xy} + i\gamma B_1 M_0.$$

Now, if the system is initially in the state $\mathbf{M} = (0, 0, M_0)$, show that this differential equation can be solved for the final magnetisation at a time T to yield

$$M_{xy}(\mathbf{x}) = i\gamma M_0 \int_0^T B_1(t) e^{-i\gamma \cdot \int_t^T \mathbf{G}(s)\cdot x(s)\,\mathrm{d}s}\,\mathrm{d}t. \quad (2)$$

If we now define a spatial frequency variable $\mathbf{k}(t) = -\gamma \int_t^T \mathbf{G}(s)\,\mathrm{d}s$, we can re-write (2) as something more pleasant. By writing the exponential factor as an integral over a three-dimensional delta function, interchanging the order of integration, and defining a new function

$$p(\mathbf{k}) = \int_0^T B_1(t)\delta^3(\mathbf{k}(t) - \mathbf{k})\,\mathrm{d}t$$

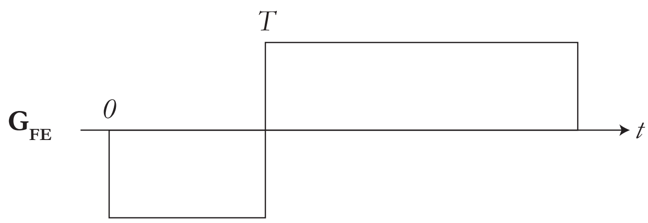
show that the magnetisation response to an RF pulse is therefore

$$M_{xy}(\mathbf{x}) = i\gamma M_0 \int_{\mathbf{K}} p(\mathbf{k}) e^{i\mathbf{x}\cdot\mathbf{k}}\,\mathrm{d}\mathbf{k} \quad (3)$$

and state what this tells you about the relationship between an RF pulse and its effect on matter.

5 Frequency Encoding Gradients

e) ** Consider a series of spins arranged on a one-dimensional lattice at locations x_i. Let the density of spins at each site be ρ_i. We are now going to explore what happens during a prephasing gradient and under a frequency encoding gradient, shown schematically below.



i) When a prephasing gradient $G_{FE,p}(t)$ is played, what will the precession frequency at each site be?

ii) In the rotating reference frame, show that the phase accrued by spin i due to this gradient is given by

$$\phi_p(x_i, T) = \gamma x_i \int_0^T G_{FE,p}(t)\,\mathrm{d}t \equiv 2\pi x_i k_{x,p}$$

where T is the duration of the prephasing gradient. Define $k_{x,p}$.

iii) The NMR signal $S_{x,p}$ observed is the sum of spin densities, with weights given by the phase. Write down this sum.

iv) If we let the number of spins become continuous, this sum generalises to

$$S_{x,p} = \int_{-\infty}^{\infty} \rho(x) e^{i\phi_p(x,\,T)}\,\mathrm{d}x$$

where $\rho(x)$ and $\phi_p(x,\,T)$ are the continuous analogies of those functions defined as above. Let us now see what happens when we play a frequency encoding gradient lobe $G_{FE}(t)$ at this point in time. Show that the NMR signal as a function of time becomes

$$S(t) = \int_{-\infty}^{\infty} \rho(x) e^{i(\phi(x,t) - \phi_p(x,\,T))}\,\mathrm{d}x \tag{4}$$

with $\phi(x,t)$ defined analogously for the frequency encoding gradient as to the prephasing gradient.

v) What happens when $(\phi(x,t) - \phi_p(x,\,T)) = 0$?

vi) As the prephasing and frequency encoding gradients have opposite signs, show that this happens at a time t_{echo} defined by

$$-\int_0^T G_{FE,p}(t)\,\mathrm{d}t = \int_0^{t_{\text{echo}}} G_{FE}(t)\,\mathrm{d}t$$

This defines the relationship between echo time and the area under gradient lobes. Additionally, (4) illustrates that the observed signal is related to the Fourier transform of spin density.